DEFORMATION OF INVOLUTION AND MULTIPLICATION IN A C^*-ALGEBRA

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Abstract. We investigate the deformation of involution and multiplication in a unital C^*-algebra when its norm is fixed. Our main result is to present all multiplications and involutions on a given C^*-algebra $\mathcal{A}$ under which $\mathcal{A}$ is still a C^*-algebra whereas we keep the norm unchanged. For each invertible element $a \in \mathcal{A}$ we also introduce an involution and a multiplication making $\mathcal{A}$ into a C^*-algebra in which a becomes a positive element. Further, we give a necessary and sufficient condition for that the center of a unital C^*-algebra $\mathcal{A}$ is trivial.

1. Introduction

A C^*-algebra is a complex Banach $*$-algebra $\mathcal{A}$ satisfying $\|a^*a\| = \|a\|^2$ $(a \in \mathcal{A})$. By the Gelfand–Naimark theorem, a C^*-algebra is a norm closed $*$-subalgebra of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. A strongly closed $*$-subalgebra of $\mathbb{B}(\mathcal{H})$ containing the identity operator is called a von Neumann algebra. By the double commutant theorem a unital $*$-subalgebra $\mathcal{A}$ of $\mathbb{B}(\mathcal{H})$ is a von Neumann algebra if and only if $\mathcal{A}$ is equal to its double commutant $\mathcal{A}^{cc}$, where $\mathcal{A}^c = \{B \in \mathbb{B}(\mathcal{H}) : AB = BA \text{ for all } A \in \mathcal{A}\}$. By Sakai's characterization of von Neumann algebras, $\mathcal{A}$ is a von Neumann algebra if and only if it is a W^*-algebra, i.e. it is a C^*-algebra being the norm dual of a Banach space $\mathcal{A}_*$. Throughout the paper $\mathcal{A}$ denotes an arbitrary C^*-algebra and $\mathcal{Z}(\mathcal{A})$ stands for its center.

For a self adjoint element $a \in \mathcal{A}$, it holds that $r(a) = \|a\|$, where $r(a)$ denotes the spectral radius of $\mathcal{A}$. This implies that the norm of a C^*-algebra is unique when we fix the involution and the multiplication. Indeed, if $\mathcal{A}$ is a C^*-algebra under two norms $\|.\|_1$ and $\|.\|_2$, then $\|a\|_1 = \|a^*a\|_1^{\frac{1}{2}} = r(a^*a)^{\frac{1}{2}} = \|a^*a\|_2^{\frac{1}{2}} = \|a\|_2$ for all $a \in \mathcal{A}$. Bohnenblust and Karlin [BK] showed that there is at most one involution on a Banach algebra with the unit 1 making it into a C^*-algebra (see also [R]): Let $*$ and $\#$ be two involutions on a unital Banach algebra $\mathcal{A}$ making it into C^*-algebras.

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Let $x \in \mathcal{A}$. It follows from the fact "an element x of a unital C^*-algebra is self-adjoint if and only if $\tau(x)$ is real for every bounded linear functional τ on $\mathcal{A}$ with $\| \tau \| = \tau(1) = 1$ ([KR 1, Proposition 4.3.3])" that x is self-adjoint with respect to $*$ if and only if it is self-adjoint with respect to $\#$. Now let $a \in \mathcal{A}$ be arbitrary and $a = a_1 + \mathrm{i}a_2$ with self-adjoint parts a_1, a_2 with respect to $*$. Then $a_1^* = a_1^\#$ and $a_2^* = a_2^\#$ and $a^* = a_1 - \mathrm{i}a_2 = a^\#$. There is another way to show the uniqueness of the involution. Indeed if $*$ and $\#$ be two involutions on a unital Banach algebra $\mathcal{A}$ making it into C^*-algebras, then the identity map from $(\mathcal{A}, *)$ onto $(\mathcal{A}, \#)$ is positive (see [P, Proposition 2.11]) and so $a^* = a^\#$ for all $a \in \mathcal{A}$.

There are several characterizations of C^*-algebras among involutive Banach algebras, see [DT] in which the authors start with a C^*-algebra and modify its structure. We however investigate a different problem in the same setting. In fact we investigate the deformation of involution and multiplication in a unital C^*-algebra when its norm is fixed. Our main result is to present all multiplications $\circ$ and involutions $\star$ on a given C^*-algebra $\mathcal{A}$ under which $\mathcal{A}$ is still a C^*-algebra whereas we keep the norm unchanged. As an application, for each invertible element $a \in \mathcal{A}$ we introduce an involution and a multiplication making $\mathcal{A}$ into a C^*-algebra in which a becomes a positive element. Further, we give a necessary and sufficient condition for that the center of a unital C^*-algebra $\mathcal{A}$ is trivial.

Recall that a Jordan $*$-homomorphism is a self-adjoint map preserving squares of self-adjoint operators. Jacobson and Rickart [JR] showed that for every Jordan $*$-homomorphism ρ of a C^*-algebra $\mathcal{A}$ with the unit 1 into a von Neumann algebra $\mathcal{B}$ there exist central projections $p_1, p_2 \in \mathcal{B}$ such that $\rho = \rho_1 + \rho_2$, $\rho(1) = p_1 + p_2$, $\rho_1(a) = \rho(a)p_1$ is a $*$-homomorphism and $\rho_2(a) = \rho(a)p_2$ is a $*$-antihomomorphism. Kadison [K] showed that an isometry of a unital C^*-algebra onto another C^*-algebra is a Jordan $*$-isomorphism.

2. Results

We start our work with the following lemma.

Lemma 2.1. *Let $\mathcal{A}$ be a unital C^*-algebra of operators acting on a Hilbert space $\mathcal{H}$. Let $p \in \mathcal{A}$ be a central projection and $u \in \mathcal{A}$ be a unitary. Let $\circ$ be the multiplication and $\star$ be the involution defined on $\mathcal{A}$ by*

$$a \circ b = paub + (1-p)bua \quad \text{and} \quad a^\star = u^* a^* u^* \tag{2.1}$$

for $a, b \in \mathcal{A}$, respectively. Then $\mathcal{A}$ equipped with the multiplication $\circ$ and the involution $\star$ is a unital C^*-algebra.

Proof. It is easy to check that $\mathcal{A}$ is a complex Banach algebra under the multiplication $\circ$ and u^* is the unit for this multiplication. By the decomposition

$\mathcal{H} = p\mathcal{H} \oplus (1-p)\mathcal{H}$, we can represent any element $a \in \mathcal{A}$ by the 2×2 matrix $\begin{pmatrix} pa & 0 \\ 0 & (1-p)a \end{pmatrix}$. For $a, b \in \mathcal{A}$, therefore $pa + (1-p)b$ can be identified by $\begin{pmatrix} pa & 0 \\ 0 & (1-p)b \end{pmatrix}$, whence $||pa + (1-p)b|| = \max(\|pa\|, \|(1-p)b\|)$. Hence

$$\begin{aligned} \|a^\star \circ a\| &= \|p\ u^*a^*a + (1-p)\ aa^*u^*\| \\ &= \max(\|pa^*a\|, \|aa^*(1-p)\|) \\ &= \max(\|pa\|^2, \|(1-p)a\|^2) \\ &= \max(\|pa\|, \|(1-p)a\|)^2 = \|a\|^2 \end{aligned}$$

for all $a \in \mathcal{A}$. $\square$

The unital C^*-algebra $\mathcal{A}$ equipped with the multiplication $\circ$ and the involution $\star$ is denoted by $\mathcal{A}(\circ, \star)$. Next we establish a converse of Lemma 2.1.

Theorem 2.2. *Let $\mathcal{A}$ be a unital C^*-algebra of operators acting on a Hilbert space $\mathcal{H}$ and there exist a multiplication $\circ$ and an involution $\star$ on the normed space $\mathcal{A}$ making it into a C^*-algebra. Then there exists a unitary element $u \in \mathcal{A}$ and a central projection p in the double commutant of $\mathcal{A}^{cc}$ of $\mathcal{A}$ such that both equalities* (2.1) *hold.*

Proof. Since $\mathcal{A}$ is unital, the closed unit ball of $\mathcal{A}$ has an extreme point, hence the C^*-algebra $\mathcal{A}(\circ, \star)$ is unital. Since $\iota(x) = x$ is an isometric linear map of $\mathcal{A}$ onto $\mathcal{A}(\circ, \star)$, the unitary elements of $\mathcal{A}(\circ, \star)$ and those of $\mathcal{A}$ coincide [KR 2, Exercise 7.6.17]. Thus if u^* is the unit of $\mathcal{A}(\circ, \star)$, then u is a unitary of $\mathcal{A}$. Define $\rho : \mathcal{A} \to \mathcal{A}(\circ, \star)$ by $\rho(a) = u^*a$. Clearly ρ is a unital isometric linear map of $\mathcal{A}$ onto $\mathcal{A}(\circ, \star)$. Hence ρ is a positive map. This implies that $u^*a^* = \rho(a^*) = (u^*a)^\star$ and so $a^\star = u^*a^*u^*$.
For determining the multiplication, define a multiplication $\diamond$ on $\mathcal{A}^{cc}$ (with respect to the original multiplication) by (2.1) as $p = 1$. Then $\mathcal{A}^{cc}$ with the multiplication $\diamond$ is a C^*-algebra. The space $\mathcal{A}^{cc}$ as a Banach space is already the dual of a Banach space, so $\mathcal{A}^{cc}$ with the new product and the new involution is a von Neumann algebra. Then the map $\rho(x) = x$ is a unital isometric linear map of $\mathcal{A}(\circ, \star)$ into the von Neumann algebra $\mathcal{A}^{cc}(\diamond, \star)$. By the result of Kadison [K] it is a Jordan $*$-isomorphism and by the Jacobson and Rickart theorem [JR] there exists a central projection p' in $\mathcal{A}^{cc}(\diamond, \star)$ such that $\rho_1(x) = p' \diamond \rho(x)$ is a $*$-homomorphism and $\rho_2(x) = (u^* - p') \diamond \rho(x)$ is a $*$-antihomomorphism. Therefore for each $a, b \in \mathcal{A}$ we

have

$$\begin{aligned}
a \circ b &= \rho(a \circ b) \\
&= \rho_1(a \circ b) + \rho_2(a \circ b) \\
&= p' \diamond \rho_1(a) \diamond \rho_1(b) + (u^* - p') \diamond \rho_2(b) \diamond \rho_2(a) \\
&= p' \diamond a \diamond b + (u^* - p') \diamond b \diamond a. \\
&= p' uaub + (u^* - p')ubua \\
&= p' uaub + (1 - p' u)bua.
\end{aligned}$$

Let $p = p' u$. Since $(p' u)^2 = p' up' u1 = p' \diamond (p' \diamond 1) = p' \diamond 1 = p' u$ and $(p' u)^* = u^* p'^* = u^* p'^* u^* u = p'^{\star} u = p' u$, so p is a projection in $\mathcal{A}^{cc}$. A similar argument shows that $\theta : \mathcal{A}^{cc}(\diamond, \star) \to \mathcal{A}^{cc}$ defined by $\theta(a) = au$ is a Jordan $*$-isomorphism. So, by [JR, Corollary 1] , $\theta(\mathcal{Z}(\mathcal{A}^{cc}(\diamond, \star))) = \mathcal{Z}(\theta(\mathcal{A}^{cc}(\diamond, \star)))$. Therefore $pa = \theta(p')\theta(au^*) = \theta(au^*)\theta(p') = ap$ for each $a \in \mathcal{A}$. Hence p is a central projection in $\mathcal{A}^{cc}$. □

Remark 2.3. Note that in general case, a C^*-algebra $\mathcal{A}$ has many representations. However the proof of Theorem 2.2 shows that for any representation of $\mathcal{A}$, we can present all multiplications and involutions on $\mathcal{A}$ which keep it still a C*-algebra with the same norm by a unitary and a central projection in the double commutant with respect to the same representation. Further, since p in Theorem 2.2 is in $\mathcal{A}^{cc} \subseteq \mathbb{B}(\mathcal{H})$, it depends on $\mathcal{H}$. If $\mathcal{A}$ is a von Neumann algebra, then $p \in \mathcal{A}^{cc} = \mathcal{A}$.

Corollary 2.4. *Let $\mathcal{I}$ be an ideal of a von Neumann algebra $\mathcal{A}$. Then $\mathcal{I}$ is also an ideal of the C^*-algebra $\mathcal{A}(\circ, \star)$ for any multiplication $\circ$ and any involution $\star$.*

Proof. It is sufficient to note that $paub$ and $(1-p)bua$ belong to $\mathcal{I}$ when $a \in \mathcal{A}, b \in \mathcal{I}$ and so $a \circ b = paub + (1 - p)bua \in \mathcal{I}$. □

It is easy to see that $a \circ b = b \circ a$ if and only if $aub = bua$. We therefore have

Corollary 2.5. *Suppose that $\mathcal{A}$ is a unital C^*-algebra and the normed space $\mathcal{A}$ is equipped with a multiplication $\circ$ and an involution $\star$ is a C^*-algebra with the unit u^*, where $u \in \mathcal{A}$ is a unitary. Then*
(i) $\mathcal{A}$ is commutative if and only if so is $\mathcal{A}(\circ, \star)$.
(ii) $\mathcal{Z}(\mathcal{A}) = \mathbb{C}1$ if and only if $\mathcal{Z}(\mathcal{A}(\circ, \star)) = \mathbb{C}u^*$.

Proof. (i) Let $\mathcal{A}$ be commutative. By Theorem 2.2 there exist a unitary element $u \in \mathcal{A}$ and a central projection p in $\mathcal{A}^{cc}$ such that

$$a \circ b = paub + (1 - p)bua \qquad (a, b \in \mathcal{A}).$$

Hence

$$a \circ b = paub + (1-p)bua = pbua + (1-p)aub = b \circ a\,.$$

Therefore $\mathcal{A}(\circ, \star)$ is commutative. Changing the role of $\mathcal{A}$ by $\mathcal{A}(\circ, \star)$, we reach the reverse assertion.

(ii) Let $\mathcal{Z}(\mathcal{A}) = \mathbb{C}1$. If $a \in \mathcal{Z}(\mathcal{A}(\circ, \star))$, then for any $b \in \mathcal{A}$ we have $a \circ b = b \circ a$. As in the proof of Theorem 2.2 we observe that $\theta : \mathcal{A}^{cc}(\circ, \star) \to \mathcal{A}^{cc}$ defined by $\theta(a) = au$ is a Jordan $*$-isomorphism. Hence, by [JR, Corollary 1] , $\theta(b)\theta(a) = \theta(a)\theta(b)$, so $aubu = buau$. Since each element of $\mathcal{A}$ is of the form bu for some $b \in \mathcal{A}$, it follows that $au \in \mathcal{Z}(\mathcal{A})$. Hence $au = \lambda 1$ for some $\lambda \in \mathbb{C}$. Therefore $\mathcal{Z}(\mathcal{A}(\circ, \star)) = \mathbb{C}u^*$. Similarly we can deduce the converse. □

Remark 2.6. The Arens product on $(c_0)^{**} = l^\infty$ coincide with the usual product in l^∞ [D, Example 2.6.22]. It was extended to arbitrary C^*-algebras in [BD]. We reprove the fact in our own way: Let $\mathcal{A}$ be a C^*-algebra and its second dual $\mathcal{A}^{**}$ be also a C^*-algebra under a multiplication $(a, b) \longmapsto a \cdot b$ whose restriction to $\mathcal{A} \times \mathcal{A}$ is the same multiplication of $\mathcal{A}$. We shall show that the Arens product (denoted by $\diamond$) on $\mathcal{A}^{**}$ is the same as the multiplication $\cdot$ on $\mathcal{A}^{**}$. It is known that $\mathcal{A}^{**}$ is a von Neumann algebra under the Arens multiplication [D, Theorem 3.2.37]. By the Kaplansky density theorem, $\mathcal{A}$ is dense in $\mathcal{A}^{**}$ in the weak*-topology, so there exists a net u_α in $\mathcal{A}$ such that $u_\alpha \to 1$ in the weak*-topology in which 1 denotes the unit of $\mathcal{A}^{**}$. So

$$b = w^* - \lim_\alpha u_\alpha b = w^* - \lim_\alpha u_\alpha \diamond b = 1 \diamond b$$

for each $b \in \mathcal{A}$. The Kaplansky density theorem implies that $1 \diamond x = x$ for each $x \in \mathcal{A}^{**}$. Therefore the units of both multiplications $\cdot$ and $\diamond$ are same. By Theorem 2.2 there exist a central projection $p \in \mathcal{A}$ such that

$$x \diamond y = pxy + (1-p)yx,$$

for each $x, y \in \mathcal{A}^{**}$. On the other hand for each $a, b \in \mathcal{A}$, we have $a \diamond b = ab$. So $(1-p)ab = (1-p)ba$. Since $\mathcal{A}$ is dense in $\mathcal{A}^{**}$ in the weak*-topology, we have $(1-p)xy = (1-p)yx$ for each $x, y \in \mathcal{A}^{**}$. Therefore $x \diamond y = pxy + (1-p)yx = pxy + (1-p)xy = xy$ for each $x, y \in \mathcal{A}^{**}$. For instance, we deduce that the Arens product on $\mathbb{K}(\mathcal{H})^{**} = \mathbb{B}(\mathcal{H})$ is equal to the operator multiplication on $\mathbb{B}(\mathcal{H})$.

Theorem 2.7. *Let $\mathcal{A}$ be a unital C^*-algebra. Then the following assertions are equivalent:*

(i) $\mathcal{Z}(\mathcal{A}) = \mathbb{C}1$

(ii) If for invertible operators $a, b \in \mathcal{A}$, $||axb|| = ||x||$ holds for each $x \in \mathcal{A}$, then there exists $\lambda > 0$ such that both λa and $\frac{1}{\lambda} b$ are unitary.

Proof. (i) $\Rightarrow$ (ii) Note that if $||a^{-1}xa|| \le ||x||$ for each $x \in \mathcal{A}$, then map $\varphi(x) = a^{-1}xa$ is a contractive unital linear map on $\mathcal{A}$. It follows from [P, Proposition 2.11] that φ is positive. Therefore $(a^{-1}xa)^* = a^{-1}x^*a$ and so $aa^*x^* = x^*aa^*$ for each $x \in \mathcal{A}$. Hence $aa^* \in \mathcal{Z}(\mathcal{A}) = \mathbb{C}1$. So $a^*a = \lambda 1$ for some $\lambda > 0$. Therefore $\frac{1}{\sqrt{\lambda}}a$ is unitary. First, assume that $||axb|| = ||x||$ for positive invertible operators a, b and each $x \in \mathcal{A}$. Then $||b^{-1}a^{-1}|| = ||a^{-1}b^{-1}|| = ||aa^{-1}b^{-1}b|| = 1$, whence

$$||a^{-1}xa|| \le ||axb||\ ||b^{-1}a^{-1}|| \le ||x||.$$

Therefore there exists $\lambda > 0$ such that $\frac{1}{\lambda}a$ is unitary. Since $\frac{1}{\lambda}a$ is positive and unitary we have $a = \lambda$. A similar argument shows that $b = \lambda'$. It follows from $1 = ||1|| = ||ab|| = \lambda'\lambda$ that $\lambda = \frac{1}{\lambda'}$. Second, assume that $||axb|| = ||x||$ for invertible operators a, b and each $x \in \mathcal{A}$. Utilizing the polar decompositions of a and b^*, there exist unitary operators u, v such that $a = u|a|$ and $b = |b^*|v$. Hence $||\ |a|\ x\ |b^*|\ || = ||u\ |a|\ x\ |b^*|\ v|| = ||axb|| = ||x||$ for each $x \in \mathcal{A}$. The above argument shows that $|a| = \lambda$ and $|b^*| = \frac{1}{\lambda}$ for some $\lambda > 0$, so $a = \lambda u$ and $b = \frac{1}{\lambda}v$.
(ii) $\Rightarrow$ (i) Note that each central invertible element a of $\mathcal{A}$ is a scalar multiple of a unitary element. In fact, we have $||a^{-1}xa|| = ||a^{-1}ax|| = ||x||$ for all $x \in \mathcal{A}$, so λa is unitary for some $\lambda > 0$. Let $a \in \mathcal{Z}(\mathcal{A})$ be a positive element and $\lambda_1, \lambda_2 \in \mathrm{sp}(a)$ are distinct. Then there exists an invertible continuous function f on $\mathrm{sp}(a)$ such that $f(\lambda_1) = \frac{1}{2}$ and $f(\lambda_2) = 1$. Hence $f(a)$, which is a central invertible element should be a scalar multiple of a unitary. On the other hand, $\frac{1}{2}, 1 \in \mathrm{sp}(f(a))$, which is impossible. Hence the spectrum of a is singleton, so $a = ||a||$. Since $\mathcal{Z}(\mathcal{A})$ is a C^*-algebra, any one of its elements is a linear combination of four positive elements. Therefore $\mathcal{Z}(\mathcal{A}) = \mathbb{C}1$. □

Let $\mathcal{A}(u, p)$ denote the C^*-algebra given via Lemma 2.1 corresponding to a unitary u and a central projection p in $\mathcal{A}$. The self-adjoint elements of $\mathcal{A}(u, p)$ are all elements a such that $au = u^*a^*$, a fact which is independent of the choice of p. Also a self-adjoint element a is positive in $\mathcal{A}(u, p)$ if and only if $a = b \circ b = pbub + (1-p)bub = bub$ for some self-adjoint element $b \in \mathcal{A}(u, p)$ and this occurs if and only if a is positive in $\mathcal{A}(u, 1)$.

Theorem 2.8. *Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}$ be invertible. Then there exists a unique unitary $u \in \mathcal{A}$ such that a is a positive element of the C^*-algebra $\mathcal{A}(u^*, p)$ for any central projection $p \in \mathcal{A}$.*

Proof. Let $a = u|a|$ be the polar decomposition of a. Then $u = a|a|^{-1} \in \mathcal{A}$. So $a = u|a|^{\frac{1}{2}}|a|^{\frac{1}{2}} = |a|^{\frac{1}{2}\star} \circ |a|^{\frac{1}{2}}$, where $\circ$ is defined in $\mathcal{A}(u^*, 1)$ by (2.1). So a is positive in $\mathcal{A}(u^*, p)$ for every central projection $p \in \mathcal{A}$. To see the uniqueness, note that if a is invertible and a, wa are positive for a unitary w, then $a = w^*(wa)$. By the

uniqueness of polar decomposition, we have $w = 1$. Now if a is positive in $\mathcal{A}(v^*, 1)$, then $a = b^\star \circ b = vb^*b$. Hence $v^*u|a| = v^*a = b^*b$ is positive. Therefore $v^*u|a|$ and $|a|$ are positive and so $v = u$ according to what we just proved. $\square$

Remark 2.9. The invertibility condition in Proposition 2.8 is essential. For example let $\mathcal{A} = C[-1,1]$ and $f(t) = t$. If f is positive in $C[-1,1](u,1)$ for a unitary function u, then there exist $g \in C[-1,1]$ such that $t = f(t) = u(t)|g(t)|^2$ for each $t \in [-1,1]$. So $u(t) = 1$ for each $t \in (0,1]$ and $u(t) = -1$ for each $t \in [-1,0)$, which is impossible.

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